

Asymptotics of a class of integrals

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Abstract

Consider an integral $I(s) := \int_0^T e^{-s(x^2 - icx)} dx$, where $c > 0$ and $T > 0$ are arbitrary positive constants. It is proved that $I(s) \sim \frac{i}{sc}$ as $s \rightarrow +\infty$. The asymptotic behavior of the integral $J(s) := \int_0^T e^{s(x^2 + icx)} dx$ is also derived. One has $J(s) \sim \frac{e^{sT^2 + iscT}}{s(2T + ic)}$ as $s \rightarrow +\infty$.

MSC: 41A60

Key words: asymptotics of integrals; complex phase function.

1 Formulation of the Result

Let $c > 0$ be a constant and $s > 0$ be a large parameter. Consider first the integral

$$I(s) := \int_0^T e^{-s(x^2 - icx)} dx, \quad (1)$$

where $T > 0$ is an arbitrary fixed constant. Our result concerning the asymptotics of the integral $I(s)$ is formulated in Theorem 1, below. The choice of $T > 0$ *does not influence* the asymptotics of $I(s)$ as $s \rightarrow \infty$. This is proved in Remark 1 below. However, the choice of T *does influence* the asymptotics of the integral

$$J(s) := \int_0^T e^{s(x^2 + icx)} dx, \quad (2)$$

as $s \rightarrow \infty$. The asymptotic behavior of the integral $J(s)$ is given in Theorem 2. This theorem is formulated below and derived in Section 3. In Section 1 one assumes without loss of generality that $T = \infty$ and uses some analytical results from [1] and [4].

The aim of this paper is to derive asymptotic formulas for the integrals $I(s)$ and $J(s)$ as $s \rightarrow +\infty$. In what follows we write ∞ for $+\infty$.

Let us formulate our first result.

Theorem 1. *One has*

$$I(s) \sim \frac{i}{sc} \quad \text{as } s \rightarrow +\infty. \quad (3)$$

In Section 2 a proof of Theorem 1 is given.

In Section 3 the asymptotic behavior of $J(s)$ is found. The following theorem is proved there.

Theorem 2. *One has*

$$J(s) \sim \frac{e^{sT^2 + iscT}}{s(2T + ic)} \quad \text{as } s \rightarrow +\infty. \quad (4)$$

In the large literature on the stationary phase and the steepest descent methods the integrals of the type $\int_D e^{i\lambda S(x)} f(x) dx$ are studied when $\lambda \rightarrow \infty$. It is usually assumed that the phase $S(x)$ is a real-valued or a purely imaginary function, that $D \subset \mathbb{R}^n$ is a bounded domain or the whole space \mathbb{R}^n , and that the phase function $S(x)$ has finitely many non-degenerate critical points in D , see, for example, [2], [3].

There are some works on the steepest descent method when the phase function $S(x)$ is assumed analytic and one chooses a steepest descent contour on which the phase function is either a real-valued or a purely imaginary function.

We study two examples in which the phase function is complex-valued. The special choice of the phase function is motivated by a study of the Pompeiu problem, see [5], Chapter 11.

Our derivations of the asymptotic formulas for the integrals $I(s)$ and $J(s)$ differ from the derivations in [2] and [3]. Namely, we use some formulas from the theory of special functions, see [1], [4], in the proof of Theorem 1, and some simple argument in the proof of Theorem 2. The principally novel point in our results is the fact that the phase functions $x^2 + icx$ and $-x^2 + icx$ are neither real-valued nor purely imaginary.

Our presentation is independent of the published works and has practically no intersection with the published literature.

2 Proof of Theorem 1

Let us start with two formulas from [1], formulas 1.4.11 and 2.4.18:

$$\int_0^\infty e^{-ax^2} \cos(xy) dx = \frac{\pi^{1/2}}{2a^{1/2}} e^{-\frac{y^2}{4a}}, \quad a > 0, \quad (5)$$

and

$$\int_0^\infty e^{-ax^2} \sin(xy) dx = \frac{ye^{-\frac{y^2}{4a}}}{2a} F(1/2; 3/2; \frac{y^2}{4a}), \quad a > 0, \quad (6)$$

where $F(b; c; x)$ is the degenerate hypergeometric function, defined in [4], Chapter 9, by the formula

$$F(b; c; x) = \sum_{k=0}^{\infty} \frac{(b)_k x^k}{(c)_k k!},$$

where $(b)_k := \frac{\Gamma(b+k)}{\Gamma(b)}$, $(b)_0 := 1$. In [1] the function $F(b; c; x)$ is denoted sometimes by ${}_1F_1(b; c; x)$.

One has ([4], formula (9.12.8)):

$$F(b; c; x) \sim \frac{\Gamma(c)}{\Gamma(b)} e^x x^{b-c} [1 + O(\frac{1}{x})], \quad x \rightarrow +\infty. \quad (7)$$

Let $y = cs$ and $a = s$, $b = 1/2$, $c = 3/2$, $x = \frac{y^2}{4s}$. It follows from formulas (5)-(7) after some simple algebraic calculations that

$$I(s) = \int_0^\infty e^{-sx^2} e^{icsx} dx \sim \frac{i}{sc}, \quad s \rightarrow \infty. \quad (8)$$

Theorem 1 is proved. □

Remark 1. If $T > 0$ is an arbitrary fixed number, then

$$\int_0^T e^{-sx^2} e^{icsx} dx \sim \frac{i}{sc}, \quad s \rightarrow \infty. \quad (9)$$

This follows from the estimate

$$|\int_\epsilon^T e^{-sx^2} e^{icsx} dx| \leq O(e^{-s\epsilon^2}) = o(s^{-1}), \quad s \rightarrow \infty, \quad (10)$$

which holds for any number $\epsilon > 0$ and any $T > \epsilon$.

Note that one can calculate some other integrals using the formula for $I(s)$. For example,

$$I_1(s) := \int_0^T e^{-sx^2} e^{icsx} x^2 dx = s^{-2} (-id/dc)^2 I(s) \sim -2is^{-3} c^{-3}, \quad s \rightarrow \infty. \quad (11)$$

where the differentiation with respect to parameter c is justified.

One may take $T = \infty$ without loss of the generality, as was mentioned above, see Remark 1.

The result (8) can be formally obtained if one neglects the term $-sx^2$ in the phase, uses standard asymptotics of the integral $\int_0^a e^{icsx} f(x) dx$ as $s \rightarrow \infty$, and assumes that $f \in C^1([0, a])$, $f = 1$ in a neighborhood of the origin $x = 0$ and $f = 0$ for $x \geq a$. This is a formal argument, and the rigorous justification of the resulting asymptotic formulas (8) and (9) is given in the proof of Theorem 1 and in Remark 1.

3 Asymptotics of $J(s)$

Consider now integral (2). The principal difference of this integral compared with the integral (1) is that the asymptotic behavior of $J(s)$ depends now on both parameters T and c . Moreover, the asymptotical formula for $J(s)$ has a different nature: the asymptotic grows exponentially and oscillates as $s \rightarrow \infty$.

Let us describe the idea of our argument. The idea is to transform the integrand e^{sx^2+iscx} into a form for which the asymptotic behavior can be easily calculated. This is done as follows. One has:

$$J(s) = e^{\frac{sc^2}{4}} \int_0^T e^{s(x+\frac{ic}{2})^2} dx. \quad (12)$$

Thus,

$$J(s) = e^{\frac{sc^2}{4} + s(T+\frac{ic}{2})^2} \int_0^T e^{s[(x+\frac{ic}{2})^2 - (T+\frac{ic}{2})^2]} dx. \quad (13)$$

After some algebraic transformations one gets

$$J(s) = e^{sT^2+iscT} \int_0^T e^{-sy(2T+ic-y)} dy, \quad y = T - x. \quad (14)$$

Denote

$$J_1(s) := \int_0^T e^{-sy(2T+ic-y)} dy. \quad (15)$$

We derive below, in Lemma 1, the asymptotic formula

$$J_1(s) \sim \frac{1}{s(2T + ic)}, \quad s \rightarrow \infty. \quad (16)$$

Combining formulas (14)-(16), one gets

$$J(s) \sim \frac{e^{sT^2 + iscT}}{s(2T + ic)}, \quad s \rightarrow \infty. \quad (17)$$

Theorem 2. *The asymptotics of $J(s)$ as $s \rightarrow \infty$ is given by formula (17).*

Let us now derive formula (16).

Lemma 1. *Formula (16) holds.*

Proof. Let $\epsilon > 0$ be a small number which is specified below. One has

$$J_1(s) = \int_0^\epsilon e^{-sy(2T + ic - y)} dy + O(e^{-s\epsilon(2T - \epsilon)}). \quad (18)$$

Here we have used the fact that the function $y(2T - y)$ is monotonically growing on the interval $[0, T]$.

Let us calculate the asymptotics of the integral

$$J_2(s) := \int_0^\epsilon e^{-sy(2T + ic - y)} dy, \quad s \rightarrow \infty.$$

Let us choose $\epsilon = \epsilon(s)$ such that $s\epsilon \rightarrow \infty$ while $s\epsilon^2 \rightarrow 0$ as $s \rightarrow \infty$. This can be done in many ways. For example, one may take $\epsilon = s^{-(0.5 + \delta)}$, where $\delta \in (0, 0.5)$. With such a choice of $\epsilon(s)$ one has

$$J_2(s) = \int_0^\epsilon e^{-sy(2T + ic - y)} dy = \int_0^\epsilon e^{-sy(2T + ic)} dy [1 + o(1)] \quad s \rightarrow \infty. \quad (19)$$

The asymptotics of the integral

$$J_3(s) := \int_0^\epsilon e^{-sy(2T + ic)} dy$$

can be calculated easily:

$$J_3(s) = \int_0^\epsilon e^{-sy(2T + ic)} dy \sim \frac{1}{s(2T + ic)}, \quad s \rightarrow \infty. \quad (20)$$

Note that $O(e^{-s\epsilon(2T - \epsilon)}) = o(s^{-1})$ as $s \rightarrow \infty$ and $\epsilon = \epsilon(s)$ is chosen, for example, so that $e^{-s\epsilon(s)T} = o(s^{-1})$. This relation holds, for example, if $\epsilon(s) = s^{-0.6}$. From (18)-(20) the conclusion of Lemma 1 follows. \square

References

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